Notes Topics in Algebraic Geometry

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1 Toric Geometry

1.1 Monoids

Definition 1.1.1. A commutative monoid is a triple $(M, +, 0)$ with $+ : M^2 \to M$ and $0 \in M$ such that

- $+$ is associative,
- $+$ is commutative,
- 0 is an identity element.

Example 1.1.2. The following are examples of monoids:

- $(\mathbb{Z}_{>0}, \cdot, 1)$
- $(\mathbb{Z}_{\geq 0}, +, 0)$

Lemma 1.1.3. *Let R be a ring. Then the functor*

$$\mathbf{Alg}_R \to \mathbf{Mon} : R \mapsto (R, \cdot, 1)$$

has a left adjoint

$$R[-] : \mathbf{Alg}_R \leftarrow \mathbf{Mon} : M \mapsto R[M]$$

Proof. We construct $R[M]$ as a ring:

As an additive group it is the linearization $R[M]$ with multiplication given by

$$\sum_i x_i m_i \cdot \sum_j y_i n_i = \sum_{i,j} x_i y_j (m_i + y_j)$$

which is well-defined by finite support of exact sequences.

Example 1.1.4. For a ring R, we have natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_R}(R[\mathbb{N}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}, (A, \cdot)) \\ &\cong A \\ &\cong \mathrm{Hom}_{\mathbf{Alg}_R}(R[x], A). \end{aligned}$$

Therefore, we conclude by the Yoneda lemma that $R[\mathbb{N}] \cong R[x]$.

1.2 Rational polyhedral cones

Definition 1.2.1. Let M be a free finitely generated Abelian group with $v_1, \ldots, v_s \in M$. The cone generated by the v_i is

$$\left\{ \sum_i r_i v_i : r_i \in \mathbb{R}_{\geq 0} \right\} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} =: M_{\mathbb{R}}.$$

We identify M with the inclusion

$$M \hookrightarrow M_{\mathbb{R}} : m \mapsto m \otimes 1.$$

A subset $\sigma \subseteq M_{\mathbb{R}}$ is a rational polyhedron (RP) cone if $\exists s \in \mathbb{N}$ and $v_1, \ldots, v_s \in M$ such that σ is of this form.

Proposition 1.2.2 (Gordon's lemma)**.** *If $\sigma \in M_{\mathbb{R}}$ is an RP cone then $\sigma \cap M$ is a finitely generated monoid.*

The generators of σ don't necessarily generate $\sigma \cap M$!

1.2.1 Duality and faces

Definition 1.2.3. For any monoid we can define

$$M^\vee = \mathrm{Hom}(M, \mathbb{Z})$$

and

$$M_\mathbb{R}^\vee = \mathrm{Hom}(M_\mathbb{R}, \mathbb{R})$$

Given a cone $\sigma \subseteq M_\mathbb{R}$ we define

$$\sigma^\vee = \{n \in M_\mathbb{R}^\vee : \forall m \in \sigma, m \cdot n = n(m) \geq 0\}$$

Lemma 1.2.4. *The dual cone $\sigma^\vee$ is an RP cone.*

$(\sigma^\vee)^\vee = \sigma$.

Definition 1.2.5. We write $\langle v_1, \ldots, v_n \rangle \subseteq M_\mathbb{R}$ to be the cone generated by the v_i, concretely:

$$\langle v_1, \ldots, v_n \rangle = \left\{ \sum_i r_i v_i : \forall i, r_i \geq 0 \right\}.$$

A face of σ is a cone of the form $\sigma \cap \langle -\tau \rangle^\vee$ for any $\tau \in \sigma^\vee$.

Lemma 1.2.6. *A face of an RP cone is an RP cone.*

Remark 1.2.7. There is a nice trick for computing duals of cones. Let $M = \mathbb{Z}^n$ and $\sigma \subseteq M_\mathbb{R} \cong \mathbb{R}^n$ a cone generated by the elements $v_1, \ldots, v_n \in M$.

We have an isomorphism $M_\mathbb{R} \cong M_\mathbb{R}^\vee$ given by $v \mapsto \langle v, - \rangle$. Take any $v \in M$, there is then a line through the origin perpendicular to v. The dual $\langle v \rangle^\vee$ corresponds to the half plane H_v corresponding to that line in which v lies.

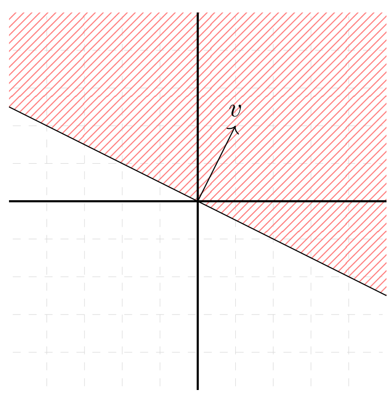


Figure 1: The dual of a cone $\langle v \rangle$.

For a general cone $\sigma = \langle v_1, \ldots, v_n \rangle$ one can then compute the dual as the intersection of the half planes:

$$\sigma^\vee = \bigcap_i H_{v_i}.$$

The right to left inclusion is easy to see: if $n \in \bigcap_i H_{v_i}$ then for all $(r_i) \in \mathbb{R}^n$

$$n \cdot \left(\sum_i r_i v_i \right) = \sum_i r_i (n \cdot v_i)$$
$$\geq 0$$

because $n \in H_{v_i}$ so $n \cdot v_i \geq 0$.

Conversely, suppose $n \in \sigma^\vee$. Then $n \cdot v_i \geq 0$ by assumption so $n \in \langle v_i \rangle^\vee = H_{v_i}$ for all i so n is also in the intersection.

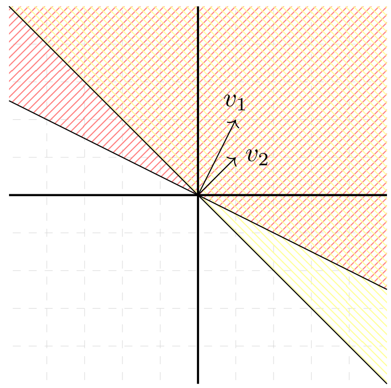


Figure 2: The dual of the cone $\langle v_1, v_2 \rangle$ is the doubly shaded region.

Definition 1.2.8. An RP cone $\sigma \subseteq M_\mathbb{R}$ is strictly convex (SRP cone) if

$$\sigma \cap -\sigma = 0$$

or equivalently σ does not contain a line through the origin.

1.2.2 Fans

Definition 1.2.9. A fan F in $M_\mathbb{R}$ is a set of SRP cones in $M_\mathbb{R}$ satisfying the following properties:

i. $\{0\} \subseteq F$,

ii. for all $\sigma \in F$ and faces τ of σ we have $\tau \in F$,

iii. for all $\sigma, \sigma' \in F$ we have that $\sigma \cap \sigma'$ is a face of σ.

If σ is an SRP cone then the set $\{\text{faces of } \sigma\}$ is a fan.

Example 1.2.10. Let $M = \mathbb{Z}^2$. Then the following is a fan:

$$F = \{\langle (1,0), (0,1) \rangle, \langle 1, 0 \rangle, \langle 0, 1 \rangle, \{0\}\}.$$

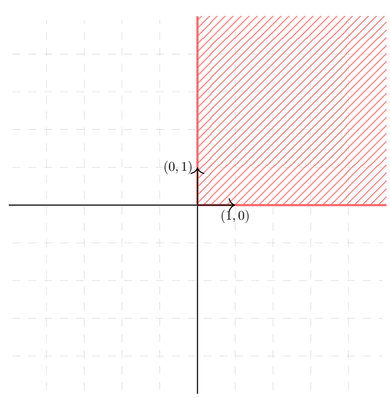


Figure 3: The fan F.

Example 1.2.11. Let $M = \mathbb{Z}$. Then

$$F = \{\sigma_2 = \langle 1 \rangle, \sigma_1 = \langle -1 \rangle, \sigma_0 = \{0\}\}$$

is a fan.

1.3 Toric schemes

We fix some ring $R \in \mathbf{CRing}$.

Definition 1.3.1. Let σ be an SRP cone. The affine toric scheme associated to σ is given by

$$X_\sigma = \mathrm{Spec}(R[\sigma^\vee \cap M^\vee]).$$

We often write $S_\sigma = \sigma^\vee \cap M^\vee$.

Remark 1.3.2. If $\sigma' \subseteq \sigma$ then $\sigma^\vee \subseteq (\sigma')^\vee$ inducing a map $S_\sigma \to S_{\sigma'}$ and therefore also a map $X_{\sigma'} \to X_\sigma$.

Example 1.3.3. If $M = \mathbb{Z}^m$ we can take

$$\sigma = R^a_{\geq 0} = \mathbb{R}^a_{\geq 0} \times \{0\}^{n-1} \subseteq M_\mathbb{R} = \mathbb{R}^n.$$

Then

$$X_\sigma = \mathbb{A}^a_R \times_R \mathbb{G}^{n-a}_{m,R}.$$

This is because $\sigma^\vee = \mathbb{R}^a_{\geq 0} \times \mathbb{R}^{n-a}$ so $S_\sigma = \mathbb{N}^a \times \mathbb{Z}^{n-a}$. We know that $R[\mathbb{N}] \cong R[x]$ and $R[\mathbb{Z}] \cong R[x, x^{-1}]$.

Exercise 1.3.4. The following are equivalent:

i. A cone σ is strictly convex,

ii. the linear span of $\sigma^\vee$ is $M^\vee_\mathbb{R}$,

iii. the map $X_{\{0\}} \to \mathrm{Spec}\, R[\sigma^\vee \cap M^\vee]$ is an open immersion[1].

Therefore, the affine schemes we get this way are those with a torus as a dense open subset.

Remark 1.3.5. The scheme $\mathrm{Spec}\, R[t, t^{-1}]$ is often called the torus.

Lemma 1.3.6 (Gluing schemes)**.** *Let I be an index set, X_i a scheme for all $i \in I$, for all i, j an open $U_{ij} \subseteq X_i$ such that $U_{ii} = X_i$ and $\varphi_{ij} : U_{ij} \cong U_{ji}$ and for all i, j, k we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ and the diagram below commutes:*

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

Then there is a scheme X and opens U_i such that $\varphi_i : X_i \cong U_i$ and

i. $X = \bigcup_i U_i$,

ii. $\varphi_i(U_{ij}) = U_i \cap U_j$,

iii. $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

[1]We write $R[S_\sigma]$ because we only defined X_s if σ is indeed strictly convex.

Lemma 1.3.7. *Let σ be an RP cone and $(\sigma') = \sigma \cap \langle \ell \rangle^\vee$ a face. Then $\sigma^\vee \subseteq (\sigma')^\vee$ and the map*

$$R[S_\sigma] \to R[S_{\sigma'}]$$

is the localization at the multiplicative subset.

$$T = \langle \ell \rangle \cap M^\vee.$$

Proof. We prove this is the localization using the universal property:

$$\begin{aligned}\mathrm{Hom}_R(R[S_{\sigma'}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(S_{\sigma'}, (A, \times)) \\ &\cong \left\{ f \in \mathrm{Hom}(S_\sigma, A) : f(\ell) \in A^\times \right\} \\ &\cong \left\{ f \in \mathrm{Hom}_R(R[S_\sigma, A]) : f(\ell) \in A^\times \right\} \\ &\cong \mathrm{Hom}_R(T^{-1}R[S_\sigma], A).\end{aligned}$$

Lemma 1.3.8. *Given a fan F we can define gluing data. We let the index set be F and define $X_\sigma = X_\sigma$ for all $\sigma \in F$. Given $\sigma, \sigma' \in F$ let $\tau = \sigma \cap \sigma'$ then define $U_{\sigma\sigma'} = X_\tau$ with isomorphism $\varphi_{\sigma\sigma'} = \mathrm{Id} : X_\tau \to X_\tau$.*

Proof. We need to verify $X_\tau \to X_\sigma$ is an open immersion and the triple overlap condition.

The overlap condition boils down to verifying the following: if τ, τ' are faces of σ then $\tau \cap \tau'$ is a face of σ and $X_{\tau\cap\tau'} = X_{\tau\cap\tau'} \subseteq X_\sigma$.

Let σ' be a face of σ. We show that $X_{\sigma'} \to X_\sigma$ is an open immersion. Let $\sigma' = \langle -\ell \rangle^\vee$ for some $\ell \in M^\vee$. Then $(\sigma')^\vee = \sigma^\vee + \langle -\ell \rangle$. The statement now follows from Lemma 1.3.7.

We leave the overlap condition to the reader.

Definition 1.3.9. Given a fan F we define X_F to be the result of the gluing data of Lemma 1.3.8. This can also be described as a colimit of affine schemes $F \to \mathbf{Sch}$: $\sigma \mapsto X_\sigma$ with the natural open immersions as morphisms.

Remark 1.3.10. Suppose F is a fan given by the faces of a single cone σ. Then $X_F = X_\sigma$.

Proof. In this case there is a natural inclusion $\tau \subseteq \sigma$ for all $\tau \in F$. Therefore, X_σ is a terminal object in the cocone of schemes given by $F \to \mathbf{Sch}$ which is then also the colimit of the diagram by general abstract nonsense.

Example 1.3.11. Consider the fan F from Example 1.2.11. Then the corresponding scheme is given by $\mathbb{P}^1_R$.

We first compute S_{σ_i} for $i = 0, 1, 2$.

For $\sigma_0 = \{0\}$ the dual cone is all of $M^\vee_\mathbb{R}$. Therefore, $S_{\sigma_0} = M^\vee_\mathbb{R} \cap M^\vee = M^\vee$.

For $\sigma_1 =$

Note that we have

$$\begin{aligned}X_{\sigma_0} &= \mathrm{Spec}\, R[\mathbb{Z}] = \mathrm{Spec}\, R[t, t^{-1}], \\ X_{\sigma_1} &= \mathrm{Spec}\, R[\mathbb{Z}_{\geq 0}] = \mathrm{Spec}\, R[t], \\ X_{\sigma_2} &= \mathrm{Spec}\, R[\mathbb{Z}_{\leq 0}] = \mathrm{Spec}\, R[t^{-1}].\end{aligned}$$

There is a pushout diagram of schemes:

$$\begin{tikzcd}
\operatorname{Spec} R[t, t^{-1}] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} R[t] \arrow[d] \\
\operatorname{Spec} R[t^{-1}] \arrow[r] & \mathbb{P}^1_R
\end{tikzcd}$$

which gives that the gluing gives $\mathbb{P}^1_R$.

Lemma 1.3.12. *Suppose we have a finite number of finitely generated groups $(M_i)_{i \leq n}$ and SRP cones $\sigma_i \subseteq (M_i)_R$. Then the product $\prod_i \sigma_i \subseteq \prod_i M_i$ is corresponds to the product of schemes $X_{\sigma_1} \times X_{\sigma_2}$.*

Proof. We note that

$$\left(\prod_i \sigma_i \right)^\vee = \prod_i \sigma_i^\vee$$

so $S_{\prod_i \sigma_i} = \prod_i S_{\sigma_i}$.

We also have

$$\prod_i S_\sigma = \bigoplus_i S_\sigma$$

because we have finitely many M_i.

We show using Yoneda that $R[\prod_i S_{\sigma_i}] \cong \bigotimes_i R[S_{\sigma_i}]$:

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{Alg}_R} \left(R \left[\prod_i S_{\sigma_i} \right], A \right) &= \operatorname{Hom}_{\mathbf{Mon}} \left(\prod_i S_{\sigma_i}, (A, \times) \right) \\
&= \operatorname{Hom}_{\mathbf{Mon}} \left(\bigoplus_i S_{\sigma_i}, (A, \times) \right) \\
&= \prod_i \operatorname{Hom}_{\mathbf{Mon}}(S_{\sigma_i}, (A, \times)) \\
&= \prod_i \operatorname{Hom}_{\mathbf{Alg}_R} \left(R \left[\sigma_i \right], A \right) \\
&= \operatorname{Hom}_{\mathbf{Alg}_R} \left(\bigotimes_i R[\sigma_i], A \right).
\end{aligned}$$

This is because tensor products are coproducts in the category of commutative R-algebras. The right adjoint functor Spec : $\mathbf{CRing}^{op} \to \mathbf{Sch}$ preserves limits, so it sends the tensor product to the product of schemes.

1.4 Toric morphisms

Now we define morphisms of toric schemes to turn the function from fans to schemes into a functor.

Definition 1.4.1. Let $f : M \to M'$ be a morphism of monoids, $\sigma \subseteq M_{\mathbb{R}}, \sigma' \subseteq M'_{\mathbb{R}}$ and f^* the pullback map $\operatorname{Hom}(M'^\vee_{\mathbb{R}}, \mathbb{R}) \to \operatorname{Hom}(M^\vee_{\mathbb{R}}, \mathbb{R})$. Then if $f[\sigma] \subseteq \sigma'$ we get a map of monoids $S_{\sigma'} \to S_\sigma$ inducing a morphism of rings $R[S_{\sigma'}] \to R[S_\sigma]$ or equivalently a scheme morphism $X_\sigma \to X_{\sigma'}$.

Remark 1.4.2. The functor $R[-]$ is not full so not every morphism between these rings is induced by monoid morphisms. Consider a map $\mathbb{Z}[\mathbb{N}] = R[x] \to \mathbb{Z}[\mathbb{N}] = R[x]$ sending $x \mapsto x^2$. This is not induced by a monoid map $\mathbb{N} \to \mathbb{N}$.

Definition 1.4.3. Let F, F' be fans on M, M' respectively and $f : M \to M'$ be a monoid morphism such that for each $\sigma \in F$ there is a $\sigma' \in F'$ such that $f[\sigma] \subseteq \sigma'$. Then f is called compatible with F and F'.

Theorem 1.4.4. *Let $f : M \to M'$ be a morphism compatible with the fans F, F'. Then it induces a morphism $X_F \to X_{F'}$. Such a map is called a toric morphism.*

Proof. First we note that for any σ there is a smallest σ' which contains it. To see this note that F^{op} is directed: if $\tau, \tau' \in F'$ then $\tau \cap \tau' \in F'$. Now because F is finite this process terminates eventually. We write σ' for the smallest cone in F' containing $\sigma \in F$.

Given any $\sigma \in F$ we know there is a map $X_\sigma \to X_{\sigma'} \to X_{F'}$. If we show that this collection of maps forms a cocone then it induces a map $X_F \to X_{F'}$ by the universal property. To prove this, by functoriality it suffices to show that the diagram

$$\begin{tikzcd} S_\tau \arrow[r] \arrow[d] & S_{\tau'} \arrow[d] \\ S_\sigma \arrow[r] & S_{\sigma'} \end{tikzcd}$$

is a commutative diagram of monoids for any $\sigma \subseteq \tau \in F$. This is immediate because the vertical maps are inclusions and the horizontal map is the restriction of the pullback f^* on the ambient space.

Proposition 1.4.5. *The map $F \mapsto X_F$ extends to a functor from fans to schemes.*

Proof. We define the functor on morphisms of fans using the morphisms defined in Theorem 1.4.4.

This functor preserves the identity because for any fan map $F \to F$ the induced map $S_\sigma \to S_\sigma$ is the identity for all $\sigma \in F$ because pullbacks preserve identity.

Similarly, this is multiplicative because the pullback preserves composition so the pullback given by a composition $F \to F' \to F''$ is the composition of the individual pullbacks.

Example 1.4.6. Let $\pi : \mathbb{Z}^n \to \mathbb{Z}$ be the projection onto the first coordinate and F a fan with support contained in $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. Let F' be the fan $F' = \left\{\{0\}, \mathbb{R}^{\geq 0}\right\}$. Then π is compatible with the fans. We therefore get a map

$$X_F \to X_{F'} \cong \operatorname{Spec} R[x] = \mathbb{A}^1_{\mathbb{R}}.$$

In particular if we take F from Example 1.2.10 then we get a morphism of rings

$$R[x] \to R[x, y] : x \mapsto R[x, y]$$

which corresponds to the natural projection $\mathbb{A}^2_R \twoheadrightarrow \mathbb{A}^1_R$.

Example 1.4.7. Let $M = \mathbb{Z}^n$ and F a collection of cones. Then $f(x) = mx$ is compatible with F for fixed $m \in \mathbb{N}$. Any cone $\sigma \in F$ is closed under scaling with a non-negative number so $f[\sigma] \subseteq \sigma \in F$. This gives an endomorphism $X_F \to X_F$.

1.5 The torus action

Definition 1.5.1. Let M be a finitely generated Abelian group of rank n and F a fan on M. Writing $T = \operatorname{Spec} R[M^\vee]$ for the torus there is a map

$$X_F \times_R T \to X_F$$

locally given for each $\sigma \in F$ by a map $R[S_\sigma] \to R[S_\sigma] \otimes_R T$ sending $x \mapsto x \otimes x$ where we embed $R[S_\sigma] \subseteq T$ by the natural inclusion $S_\sigma \to M^\vee$.

Proof. We show that this map exists. Notice that $(X_\sigma \times_R T)_{\sigma\in F}$ is an open cover of the fibre product. Therefore, if the maps are compatible on this affine cover it defines a scheme morphism.

Let $\tau \subseteq \sigma$ be cones in F. To show compatibility of the cone map means showing commutativity of the following diagram:

$$\begin{tikzcd} R[S_\sigma] \arrow[r] \arrow[d] & R[S_\sigma] \otimes T \arrow[d] \\ R[S_\tau] \arrow[r] & R[S_\tau] \otimes T \end{tikzcd}$$

This is trivial because the map into the tensor product is a natural transformation between functors and the vertical maps are both the natural localization map.

Theorem 1.5.2. *Let $M \to M'$ be a map of monoids such that F, F' are compatible with it. Then the torus action is compatible with the induced $X_F \to X_{F'}$. Concretely the diagram*

$$\begin{tikzcd} X_F \times T_M \arrow[r] \arrow[d] & X_{F'} \times T_{M'} \arrow[d] \\ X_F \arrow[r] & X_{F'} \end{tikzcd}$$

commutes.

Proof. It is sufficient the diagram commutes locally. Concretely we must have that the following commutes:

$$\begin{tikzcd} R[S_\sigma] \arrow[r] \arrow[d] & R[S_{\sigma'}] \arrow[d] \\ R[S_\sigma] \otimes R[M] \arrow[r] & R[S_{\sigma'}] \otimes R[M'] \end{tikzcd}$$

This is clearly true.

Remark 1.5.3. Given a toric scheme X_F over R. The toric action gives an action $X_F(R) \times T(R) \to X_F(R)$ of the torus rational points on the rational points of X_F which is compatible with toric morphisms.

1.6 Flat morphisms

Definition 1.6.1. Let A, B be two commutative rings and $f : A \to B$ a ring morphism. Then f is called flat if B is a flat A-module by this map.

Example 1.6.2. The localization map $R \to S^{-1}R$ is flat for any ring R and multiplicatively closed subset $S \subseteq R$.

1.7 Proper morphisms

Definition 1.7.1. We quickly define a couple properties of morphisms. Let $f : X \to Y$ be a morphism of schemes

i. f is called separated if the diagonal map $X \to X \times_Y X$ is a closed immersion,

ii. f is called of finite type if for all affine opens $V \subseteq Y$ the inverse image $f^{-1}(V)$ is quasi-compact and for all affines $U \subseteq f^{-1}(V)$ the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ turns $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_Y(V)$,

iii. f is called universally closed when any pullback along $Z \to Y$ has that $X \times_Y Z \to X$ is a closed embedding of topological spaces,

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

iv. A morphism is proper if it has all three of the above properties.

Example 1.7.2. Most "natural" morphisms of schemes are separated and of finite type. However, not all nice maps are universally closed. The standard example is the map $\mathbb{A}^1 \to \mathbb{Z}$. Pulling back along $\mathbb{A}^1 \to \mathbb{Z}$ gives the projection map $\mathbb{A}^2 \to \mathbb{A}^1$ which is not closed because the closed $Z(xy-1) \subseteq \mathbb{A}^2$ is mapped onto the open $D(x) \subseteq \mathbb{A}^1$.

Definition 1.7.3. We write $|F| = \bigcup_{\sigma \in F} \sigma \subseteq M_{\mathbb{R}}$ and call it the support of a fan.

We are going to work towards the following theorem:

Theorem 1.7.4. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. Then the induced $f : X_F \to X_{F'}$ of toric schemes is proper if and only if $f^{-1}[|F'|] = F$.*

Example 1.7.5. Consider the fan from Example 1.2.10 and the following fan

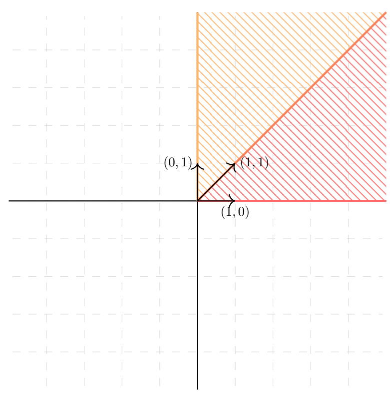


Figure 4: A "refinement" F'.

The identity on $\mathbb{Z}^2$ is compatible with $F' \to F$. The only points of $M_{\mathbb{R}}$ mapped into $|F|$ are those already in $|F'|$. Therefore, this defines a proper map of toric schemes (later we will see this corresponds to a blow-up).

Example 1.7.6. A non-example we have already seen is sending the fan $\{\langle 1 \rangle\} \subseteq \mathbb{R}$ to the zero fan in 0. It corresponds to the structure map $\mathbb{A}^1_R \to R$ which is not proper because $-1 \in \pi^{-1}[0]$.

Definition 1.7.7. Let K be a field. A discrete valuation is a map $v : K \to \mathbb{Z} \cup \{\infty\}$ with the following properties:

i. $v(x) = \infty$ if and only if $x = 0$,

ii. $v(xy) = v(x) + v(y)$,

iii. $v(x+y) \geq \min(v(x), v(y))$.

We call the ring $R = \{x \in K : v(x) \geq 0\}$ the discrete valuation ring (DVR) with valuation $v : R \to \mathbb{N} \cup \{\infty\}$.

Example 1.7.8. Let k be a field. Then the ring $k[x]_{(x)}$ is a discrete valuation ring with valuation

$$v(f) = \max\left\{i \in \mathbb{N} : x^i \mid f\right\}.$$

It has fraction field $k(x)$ the fraction field of $k[x]$.

Theorem 1.7.9 (Valuative criterion of properness)**.** *Let $X \to Y$ be a quasi-separated morphism of finite type*[2] *of Noetherian schemes*[3] *Then it is proper if and only if for all discrete valuation rings R the following commutative square has a unique lift:*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

Now we can start the proof of Theorem 1.7.4. We will prove both implications in separate lemmas.

Lemma 1.7.10. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If the induced $f : X_F \to X_{F'}$ of toric schemes over R is proper then $f^{-1}[|F'|] = F$.*

Proof. Suppose $X_F \to X_{F'}$ is proper, but there is a $w \in M \setminus |F|$ that has $f(w) \in |F'|$. We want to arrive at a contradiction: find some square such that Theorem 1.7.9 has no lift. To do this we fix some $\sigma' \in F'$ with $f(w) \in \sigma'$.

We want to find some field k and a map $R \to k$. This is purely to construct a DVR for the valuative criterion, the choice does not matter. We fix a maximal ideal $\mathfrak{m} \lhd R$ and set $k = R/\mathfrak{m}$.

We construct a commutative square by giving maps $\operatorname{Spec} k(x) \to X_F$ and $\operatorname{Spec} k[x]_{(x)} \to X_{F'}$:

$$\begin{aligned} \lambda_w : \operatorname{Spec} k(x) &\to \operatorname{Spec} R[M^\vee] \subseteq X_F \\ k(x) &\leftarrow R[M^\vee] \\ x^{u(w)} &\leftarrow\!\shortmid u \in M^\vee, \\ \lambda_{f(w)} : \operatorname{Spec} k[x]_{(x)} &\to \operatorname{Spec} X_\sigma \subseteq X_{F'} \\ k[x]_{(x)} &\leftarrow R[\sigma^\vee \cap M^\vee] \\ x^{u(f(w))} &\leftarrow\!\shortmid u \in \sigma^\vee \cap M^\vee. \end{aligned}$$

This gives a commutative square with lift by assumption of properness. We track in which affine open X_σ the point $(x) \lhd \mathbb{Q}[x]_{(x)}$ lands, and obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k(x) \arrow[r, "\lambda_w"] \arrow[d] & X_\sigma \arrow[d] \\ \operatorname{Spec} k[x]_{(x)} \arrow[r, "\lambda_{f(w)}"] \arrow[ur, dashed, "\exists!"] & X_{F'} \end{tikzcd}$$

Note that $w \notin \sigma$. This means that there is some $u \in \sigma^\vee \cap M^\vee$ with $u(w) < 0$. We obtain a commutative triangle of rings

$$\begin{tikzcd} k(x) & R[\sigma^\vee \cap M^\vee] \arrow[l] \arrow[dl] \\ k[x]_{(x)} \arrow[u] & \end{tikzcd}$$

but u is mapped to $x^{u(w)} \in k(x)$ which has $v(x^{u(w)}) = u(w) < 0$ so u cannot be sent into $k[x]_{(x)}$. This is a contradiction so no such w can exist, proving the lemma.

Now we prove the opposite implication, first over $\mathbb{Z}$ and extend our results using results about base changes of proper maps. We do this to work with Noetherian schemes for which the weakened valuative criterion is sufficient.

[2] All toric morphisms are separated and of finite type, so these restrictions are not relevant to us.

[3] There is a more general version of this for non-Noetherian schemes with all valuation rings. We need DVR's because we can construct maps to $\mathbb{Z}$ from them which live in dual spaces.

Lemma 1.7.11. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If $f^{-1}[|F'|] = F$ then the induced $f : X_F \to X_{F'}$ of toric schemes over $\mathbb{Z}$ is proper.*

Proof. We show that the induced map $X_F \to X_{F'}$ is proper using the valuative criterion of Noetherian schemes. Toric schemes over $\mathbb{Z}$ are Noetherian because affine toric schemes correspond to finitely generated $\mathbb{Z}$-algebras and therefore any toric scheme over $\mathbb{Z}$ can be covered by finitely many affine Noetherian schemes.

Let (R, v) be a DVR with fraction field K such that we have a commutative square. We want to find a lift and show its uniqueness.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X_F \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] & X_{F'}
\end{tikzcd}$$

We can assume that the top map $\operatorname{Spec} K \to X_F$ factors through the torus because X_F is irreducible[4,5].

Consider any $\sigma' \in F'$ be such that $\operatorname{Spec} R \to X_F$ factors through $X_{\sigma'}$. Then we get a commutative square of rings

$$\begin{tikzcd}
K & \mathbb{Z}[M^\vee] \arrow[l, "\alpha"'] \\
R \arrow[u] & \mathbb{Z}[\sigma'^\vee \cap M'^\vee] \arrow[l] \arrow[u, "f^*"]
\end{tikzcd}$$

Because $M^\vee$ is a group, the map $M^\vee \to K$ takes values in the units of K. This means that we obtain a map $v \circ \alpha \circ f^* : \sigma'^\vee \cap M'^\vee \to \mathbb{Z}$, it factors through R where all elements have non-negative valuation: we have a map $v \circ \alpha \circ f^* : \sigma'^\vee \cap M'^\vee \to \mathbb{N}$. This corresponds to some point in $\sigma' \cap M'$ which we will denote $v \circ \alpha \circ f^*$. It is in the image of f: it is given by $f(v \circ \alpha)$. By the assumption on f this means that there is a cone $\sigma \in F$ with $v \circ \alpha \in \sigma$. Now we fix a particular cone $\sigma' \in F'$ such that $f[\sigma] \subseteq \sigma'$ to obtain a commutative diagram

$$\begin{tikzcd}
K & \mathbb{Z}[M^\vee] \arrow[l, "\alpha"'] & \mathbb{Z}[\sigma^\vee \cap M^\vee] \arrow[l] \\
R \arrow[u] & \mathbb{Z}[\sigma'^\vee \cap M'^\vee] \arrow[l] \arrow[u, "f^*"] \arrow[ur]
\end{tikzcd}$$

Because $v \circ \alpha$ is a point in σ the map $v \circ \alpha : \sigma^\vee \cap M^\vee \to \mathbb{Z}$ takes values in $\mathbb{N}$ and therefore gifts a lift of the diagram. This shows the existence part of the valuative criterion.

Now we show uniqueness. If we show that $X_F \to X_{F'}$ is separated, we get the uniqueness of such lifts for free by the valuative criterion of separatedness. Any toric scheme over $\mathbb{Z}$ is separated[6]. We now make use of the following properties of separated morphisms: if $X \to Y \to Z$ is separated then $X \to Y$ is separated. This gives immediately that the map is separated and thus uniqueness of the lift.

We conclude that $X_F \to X_{F'}$ satisfies the valuative criterion for properness and is therefore proper.

Now we can prove the remaining implication Theorem 1.7.4 for arbitrary rings using base change.

Corollary 1.7.12. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If $f^{-1}[|F'|] = F$ then the induced $f : X_F \to X_{F'}$ of toric schemes over any ring R is proper.*

[4] The toric scheme X_F is irreducible because $\mathbb{Z}[M^\vee]$ is a dense irreducible open.

[5] This is not an easy thing to see, and the proof involves some complicated scheme theory. We refer to `https://mathoverflow.net/questions/68648/valuation-criterion-of-properness-irreducible-varieties`.

[6] This is true for a toric scheme over any ring. We will not show this but refer to any standard text e.g. Theorem 3.1.5 of Toric Varieties by Cox, Little, and Schenck.

Proof. Let R be any ring. We will write X_F for the toric scheme over $\mathbb{Z}$ and X_F^R for the scheme over R.

Suppose we satisfy all hypotheses, then $X_F \to X_{F'}$ is a proper map. We show that $X_F^R \to X_{F'}^R$ is a proper map by exhibiting it as a base change. Proper maps are preserved by base change, so the resulting map is also proper.

We can give X_F^R is given by a pullback diagram

$$\begin{tikzcd} X_F^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\ X_F \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Now we can construct the morphism we are interested in as a pullback:

$$\begin{tikzcd} X_F^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{F'}^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\ X_F \arrow[r] & X_{F'} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

This concludes the proof.

1.8 Blow-ups of toric varieties

1.8.1 Defining blow-ups

Definition 1.8.1. An irreducible divisor D in a scheme X is an irreducible closed subscheme of codimension 1 which is locally

Definition 1.8.2. Let X be a scheme and $\mathcal{I}$ a coherent sheaf. The blow-up $\widetilde{X}$ is a scheme with $\pi : \widetilde{X} \to X$ a morphism such that $\pi^{-1}\mathcal{I}$ is invertible and $\widetilde{X}$ is terminal in the full subcategory of the slice category of morphisms with this property.

It is concretely computed by projectivizing a graded algebra: $\operatorname{Proj} \bigoplus_n \mathcal{I}^n$.

Remark 1.8.3. Given a closed immersion $\imath : Z \to X$ we can blow up along $\imath_* \mathcal{O}_Z$ to obtain a blow-up $\widetilde{X} \to X$ called the blow-up in $\mathbb{Z}$.

1.8.2 Blowing up toric varieties

Definition 1.8.4 (Refinement)**.** Let M be a torsion free, finitely generated, Abelian group and F, F' two fans. We say F' refines F if

i. Each $\sigma' \in F'$ is contained in a $\sigma \in F$,

ii. the fans have the same support: $|F| = |F'|$.

Refinements give proper morphisms of toric schemes $X_{F'} \to X_F$.

Definition 1.8.5 (Star subdivision)**.** Let F be a fan of M and $\sigma = \langle u_1, \ldots, u_n \rangle \in F$ a smooth cone such that $u_1, \ldots, u_n$ is a basis of M. Let $u_0 = \sum_{i \geq 1} u_i$ and $F'(\sigma)$ the set of cones generated by subsets of $\{u_0, \ldots, u_n\}$ not containing by $\{u_1, \ldots, u_n\}$. Then

$$F^*(\sigma) = F \setminus \{\sigma\} \cup F^{'}(\sigma)$$

is called the star subdivision of F.

1.9 Divisors

We will only be considering Noetherian, integral, normal schemes.

Definition 1.9.1 (Prime divisor)**.** Let X be a scheme. A prime divisor $D \subseteq X$ is a closed, irreducible subscheme of codimension 1. We write $\operatorname{div} X$ for the free Abelian group generated by the prime divisors.

Definition 1.9.2. Let $D \subseteq X$ be a prime divisor. We will write $\mathcal{O}_{X,D}$ for the stalk $\mathcal{O}_{X,\zeta}$ of the generic point $\eta \in D$. It is a discrete valuation ring $\mathcal{O}_{X,D} \subseteq K(X)$ with valuation ν_D.

Definition 1.9.3. We have a map $K(X)^\times \xrightarrow{\operatorname{div}} \operatorname{div} X$ given by

$$f \mapsto \sum_D \nu_D(f) D.$$

We define $\operatorname{div}_0 = \operatorname{im} \operatorname{div}$ to be the principal divisors and take the quotient group $\operatorname{Cl} X = \operatorname{div} X / \operatorname{div}_0 X$.

Definition 1.9.4 (Cartier divisor)**.** Let $U \subseteq X$ be an open. Then we have a restriction $\operatorname{div} X \to \operatorname{div} U$ defined by $D \mapsto D \cap U$ (extended by 0 if $D \cap U = \varnothing$).

We then define

$$\operatorname{cdiv} X = \{D \in \operatorname{div} X : \text{There exists an open cover } \{U_i\}_i \text{ and } f_i \in K(X) \text{ with } D|_U = \operatorname{div} f_i|_{U_i}\}$$

called the Cartier divisors.

We get an exact sequence

$$0 \to \operatorname{div}_0 X \to \operatorname{cdiv} X \to \operatorname{Pic} X.$$

Proposition 1.9.5. *The following are true:*

i. *If X is smooth then* $\operatorname{Pic} X \cong \operatorname{Cl} X$,

ii. $\operatorname{Cl} \mathrm{UFD} = 0$,

iii. *There is an exact sequence*

$$0 \to \mathbb{Z}D \to \operatorname{div} X \to \operatorname{div} X \setminus D \to 0.$$

Corollary 1.9.6. *The Picard group of projective space is $\mathbb{Z}$ and generated by the hyperplane.*

Proof. Apply excision to the hyperplane by taking quotients by principal divisors which is fine.

Definition 1.9.7. Given a toric scheme X_F on a lattice M we define the characters to be the morphisms of group schemes

$$\operatorname{Hom}_{\mathbf{GrpSch}}(T, \mathbb{G}^1).$$

This turns out to be equivalent to a map $M^\vee \to \mathbb{Z}$: we have an isomorphism of groups

$$\begin{aligned} M &\to \operatorname{Hom}_{\mathbf{GrpSch}}(T, \mathbb{G}^1) \\ m &\mapsto \chi^m \end{aligned}$$

Definition 1.9.8. We write $F(1)$ for the set of rays in F.

Theorem 1.9.9. *There is a map $F(1) \to \operatorname{div} X$ defined on the standard opens X_σ by*

$$\rho \mapsto V(\{\chi^m : m \in \rho^\perp \cap S_\sigma\}) \subseteq X_\sigma$$

giving a decomposition

$$X_F \setminus T = \bigcup_\rho D_\rho.$$

Example 1.9.10. Given the affine space $\mathbb{A}^2$ given as a toric scheme by $\sigma = \mathbb{N}^2 \subseteq \mathbb{Z}^2$ the rays are the two axes.

The complement $\mathbb{A}^2 \setminus \mathbb{G}^2$ is exactly given by the two axes of $\mathbb{A}^2$ which is the union of the desired divisors.

Remark 1.9.11. We can see $\chi^m : T \to \mathbb{G}^1$ defines a function $\chi^m \in K(X)^\times$ which makes $\operatorname{div} \chi^m$ well-defined.

Proposition 1.9.12. *There is an equality*

$$\operatorname{div} \chi^m = \sum_\rho \langle u_\rho, m \rangle \, D_\rho$$

where u_ρ is the minimal generator of ρ.

Proof. The function χ^m cannot have poles or roots on T because it maps into $\mathbb{G}$.

We apply excision to get an exact

$$0 \to \bigoplus_\rho \mathbb{Z} D_\rho \to \operatorname{div} X_F \to \operatorname{div} T \to 0$$

by excision. Because χ^m is zero on T this means that it is a sum of elements in $\bigoplus_\rho \mathbb{Z} D_\rho$.

By coordinate transformation we can assume $u_\rho = e_1 \in M$. We identify the character with a monomial

$$\chi^m \simeq x^{\left\langle m, u_{\rho_1} \right\rangle} \dots x^{\langle m, u_{\rho_n} \rangle}$$

from which we get the desired statement.

Theorem 1.9.13. *There is an exact sequence*

$$M \xrightarrow{m \mapsto \operatorname{div} \chi^m} \bigoplus \mathbb{Z} D_\rho \to \operatorname{Cl} X_F \to 0.$$

1.10 Chow groups and intersection theory

Example 1.10.1. Let X be a real differential manifold of dimension n and an integer $k \geq 0$. Then we have cohomology groups $H_k(X; \mathbb{Z})$ and a cup product

$$H_{n-k}(X; \mathbb{Z}) \times H_{n-l}(X; \mathbb{Z}) \to H_{n-k-l}(X; \mathbb{Z})$$

corresponding to taking the "intersection" of submanifolds.

We try to generalize this to schemes where we will get an isomorphism

$$A_k(X) \cong H_{2k}(X; \mathbb{Z})$$

where on the right-hand side we give $X/\mathbb{C}$ the analytic topology.

Definition 1.10.2 (Chow group)**.** Take X a scheme of finite type and $k \geq 0$. We define

$$Z_k(X) = \left\{ \sum_i u_i [v_i] : u_i \in \mathbb{Z}, v_i \text{ is a closed subvariety of dimension } k \right\}.$$

For every $k+1$-dimensional subvariety ω of X, $f \in \kappa(\omega)$ we define

$$\operatorname{div}_\omega f = \sum_V \operatorname{order}_V(\varphi)[V] \in Z_k(X).$$

We define $B_k(X) \subseteq Z_k(X)$ to be the subgroup generated by all cycles of the form $\operatorname{div}_\omega f$. The Chow group is then defined as

$$A_k(X) = Z_k(X)/B_k(X).$$

We also define

$$Z_*(X) = \bigoplus_k Z_k(X),$$
$$A_*(X) = \bigoplus_k A_k(X).$$

Definition 1.10.3. Let $\imath : Y \hookrightarrow X$ be a closed subscheme. Then we define

$$\begin{aligned} \imath_*{:} A_k(Y) &\to A_k(X) \\ [Z] &\mapsto [Z]. \end{aligned}$$

Lemma 1.10.4. *Given $Y \subseteq X$ a closed subscheme and $U = Y \setminus X$ we get an exact sequence*

$$A_k(Y) \to A_k(X) \to A_k(Y) \to 0.$$

Proposition 1.10.5. *The Chow groups of $\mathbb{A}^n$ are given by*

$$A_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for projective space we have

$$A_k(\mathbb{P}^n) \cong \mathbb{Z}$$

for $0 \leq k \leq n$.

Remark 1.10.6. We have an intersection pairing on $\mathbb{P}^n$

$$\begin{aligned} A_{n-k}(\mathbb{P}^n) \times A_{n-l}(\mathbb{P}^n) &\to A_{n-k-l}(\mathbb{P}^n) \\ (a, b) \in \mathbb{Z}^2 &\mapsto ab \in \mathbb{Z}. \end{aligned}$$

If $\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$ then $[X \cap Y] = [X] \cdot [Y]$. We might have to choose representatives such that they intersect transversally for this to be well-defined.

Definition 1.10.7. Let $f : X \to Y$ be a proper morphism and $Z \subseteq X$ a subvariety. Then $f(Z) \subseteq Y$ is a subvariety of dimension at most $\dim Z$ giving a pushforward

$$f_*([Z]) = \begin{cases} ([\kappa(Z) : \kappa(f(Z))]) \cdot [f(Z)] & \dim f(Z) = \dim \mathbb{Z} \\ 0 & \text{else.} \end{cases}$$

1.10.1 Chow groups on toric varieties

Remark 1.10.8. On a toric variety we always have that

$$A_{n-1} = \operatorname{div}(X)/\kappa(X)^\times = \operatorname{Cl} X$$

Theorem 1.10.9. *Let X be a toric variety of a fan F. Then $A_k(X)$ is generated by the orbit closures $v(\sigma)$ of the cones σ of dimension $n - k$ of F.*

Proof. We define $X_i = \bigcup_{\sigma, \dim \sigma \geq n-i} v(\sigma)$ giving inclusions $X_i \supseteq X_{i-1}$. Then $X_i \setminus X_{i-1}$ is exactly the orbits of cones of dimension $n-i$ written as $\bigcup_{\sigma, \dim \sigma = n-1} O_\sigma$.

This gives an exact sequence

$$A_k(X_{i-1}) \to A_k(X_i) \to \bigoplus_{\dim \sigma = n-i} A_k(O_\sigma) \to 0.$$

Now the O_σ are the tori T_{N/N_σ} which are opens in $\mathbb{A}^i$. Therefore, excision gives

$$A_k(T) = \begin{cases} \mathbb{Z} & k = i \\ 0 & \text{else.} \end{cases}$$

We have an exact sequence

$$A_k(X_{i-1}) \to A_k(X_i) \to \bigoplus_{\dim \sigma = n-i} \mathbb{Z}[O_\sigma] \to 0.$$

By induction, we may assume that

$$A_k(X_i) = \begin{cases} 0 & k > i \\ \bigoplus_{\dim \sigma = n-i} \mathbb{Z}[O_\sigma] & \text{else,} \end{cases}$$

for $k \leq i$.

1.11 Blowing up in a monomial ideal

All toric varieties that occur in this talk are over some field k.

1.11.1 Blowing up a ring

We first inspect what blow-ups of affine schemes look like. We do this by first considering ideals that are already invertible.

Lemma 1.11.1. *Let R be a ring, $f \in R$ regular and $I = (f)$ generated*[7] *by $r_1, \ldots, r_n$. Then there are $f_1, \ldots, f_n$ such that r_i generates $I_{f_i} \triangleleft R_{f_i}$ and $(f_1, \ldots, f_n) = R$.*

Proof. Let $f \in R$ be a generator of I. Then for each i there is an $f_i \in R$ such that $r_i = f_i f$. We claim these have the desired property. Clearly $(f) = (r_i)$ in R_{f_i} because $f_i^{-1} r_i = f$ in R_{f_i}. Therefore, all we need to do is show that $(f_1, \ldots, f_n) = R$.

Now suppose that $(f_1, \ldots, f_n) \neq R$, then there is a maximal ideal $\mathfrak{m} \triangleleft R$ such that $(f_1, \ldots, f_n)_\mathfrak{m} \to R_\mathfrak{m}$ is not surjective. This means that $f_1, \ldots, f_n \in \mathfrak{m}$, else one of them would have been a unit which generates $R_\mathfrak{m}$. Therefore, we must have that

$$(f) = (r_1, \ldots, r_n) \subseteq (f_1, \ldots, f_n)(f) \subseteq \mathfrak{m}(f) \subseteq (f)$$

giving $\mathfrak{m}(f) = (f)$ in the local ring $R_\mathfrak{m}$. Nakayama's lemma then gives $(f) = 0$. Because f was regular this gives a contradiction.

This shows that for a set of generators $r_1, \ldots, r_n$ of a nice enough invertible ideal, we can find an affine open cover such that one of these elements generates the ideal on each open. We emulate this property to get a definition of the blow-up of an affine scheme:

[7] This is obviously not the smallest set of generators.

Definition 1.11.2 (Blow-up)**.** Let R be a ring and $I = (r_1, \ldots, r_n) \triangleleft R$ an ideal. We can construct the blow-up of $\operatorname{Spec} R$ in I to be the scheme constructed in the following manner:

For each r_i construct the ring
$$R_i = \frac{R[x_j]_{j \neq i}}{(r_j - xr_j)} = R\left[\frac{r_j}{r_i}\right].$$

Then clearly $(r_1, \ldots, r_n) = (r_i)$ in R_i. These rings will cover the blow-up. We glue R_i and R_j along the localization
$$R_{ij} = R\left[\frac{r_j}{r_i}, \frac{r_i}{r_j}\right] = (R_i)_{x_j} = (R_j)_{x_i}.$$

This defines a scheme which is the blow-up $\operatorname{Bl}_I R$ in the ideal I. We also get the blow-up map $\operatorname{Bl}_I R \to \operatorname{Spec} R$ from the natural inclusion maps $R \to R_i$.

The toric varieties we consider are normal, but blow-ups are not. Therefore, we will consider the normalized blow-up.

1.11.2 Normalization

Definition 1.11.3 (Integral elements and integral closure)**.** Let R be an integral domain with fraction field K. We call an element $a \in K$ integral over R if there exists some monic polynomial $f \in R[x]$ such that $f(a) = 0$.

A domain is integrally closed if it contains all elements which are integral over it.

Lemma 1.11.4. *The integral closure has a few nice properties. We will give a few*

i. Given any domain, there is a smallest integrally closed ring containing it:

$$\bigcap_{R \subseteq A \subseteq K \text{ integrally closed}} A.$$

It is called the integral closure.

ii. Given integrally closed rings $(R_i)_i$ with fraction field K, the intersection $\bigcap_i R_i$ is also integrally closed.

iii. For any domain R the following are equivalent:

(a) R is integrally closed,

(b) $R_\mathfrak{p}$ is integrally closed for all prime ideals $\mathfrak{p} \triangleleft R$,

(c) $R_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m} \triangleleft R$.

iv. Localizations of integrally closed rings are integrally closed.

v. Let $R \subseteq R'$ be two domains such that $Q(R) = Q(R') = K$ and R' is integrally closed. If the integral closure of R contains R' then it is R'.

Definition 1.11.5. A scheme X is normal if all of its stalks $O_{X,x}$ are integrally closed domains.

Theorem 1.11.6. *Given a scheme X there is a scheme normal scheme X^ν and morphism $X^\nu \to X$ such that locally this morphism is given by inclusion into the integral closure.*

Theorem 1.11.7. *Toric varieties over a field are normal.*

Proof. We show that any toric varieties has a cover of integral domains.

First we prove that the toric variety of a ray is normal. Let ρ be a ray in $M \otimes R$. Without loss of generality we may assume that it is the positive part of the first axis. Then $X_\rho = k[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ which is the localization of an integrally closed domain and hence integrally closed.

Now any cone σ is generated by its rays $\rho_1, \dots, \rho_m$. This means that $\sigma^\vee = \bigcap_i \rho_i^\vee$, so also $k[S_\sigma] = \bigcap_i k[S_{\rho_i}]$ which is the intersection of integrally closed domains and therefore integrally closed. This gives that the standard affine opens are normal.

1.11.3 Computing blow-ups of toric varieties

Now we are ready for describing the process of computing normalized blow-ups of affine toric varieties in monomial ideals. We will only give the process for blowing up a monomial ideal of $\mathbb{A}_k^2$ generated by two monomials.

Let $(X^{a_1}Y^{b_1}, X^{a_2}Y^{b_2})$ be a monomial ideal of $k[x, y]$. We can identify this ideal with points of $\mathbb{Z}^2$ as in Figure 5. We also pick the monomials to the right and above the points because these are also in the ideal.

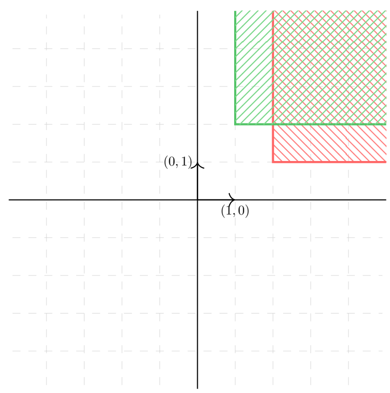


Figure 5: Drawing the ideal.

Next we take the convex hull of this shape

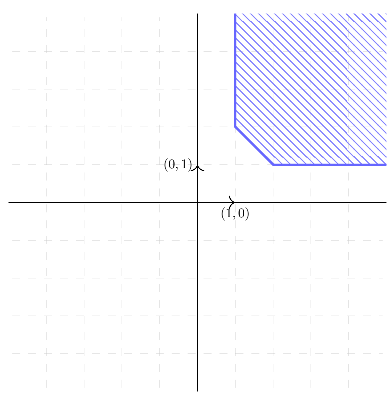
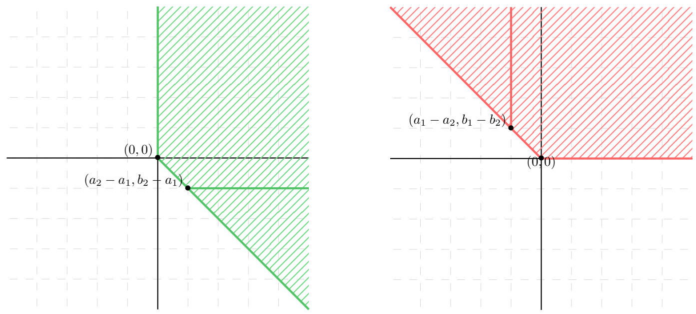


Figure 6: Drawing the convex hull of the ideal.

We translate all corners to the origin and look at the cones they generate.



(a) The first translation $\sigma^\vee$. (b) The second translation $\sigma'^\vee$.

Figure 7: Shifting both corners of the convex hull to the origin.

Taking the dual cones of these we get a fan, which is the normalized blow-up:

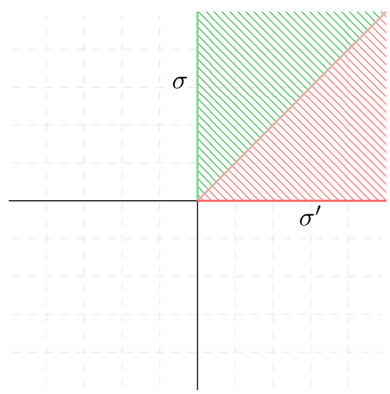


Figure 8: The fan of the normalized blow-up.

Lemma 1.11.8. *Let $\sigma \subseteq M$ be a cone with rays $\rho_1, \ldots, \rho_n$ and $x_1, \ldots, x_n$ points on $\rho_1, \ldots, \rho_n$ respectively. Then writing $\langle (x_i)_i \rangle_{\mathbb{N}}$ for the monoid generated by these points, the ring $k[\sigma \cap M]$ is integral over $k[\langle (x_i) \rangle_{\mathbb{N}}]$.*

Proof. Take any point $z \in \sigma \cap M$. We show it is integral over A. Because it is in the cone it is a $\mathbb{Q}_{\geq 0}$-linear combination of the x_i:

$$z = \sum_i \frac{p_i}{q_i} x_i.$$

Therefore, $\sum_i p_i \prod_{j \neq i} q_i x_i \in \langle (x_i)_i \rangle_{\mathbb{N}}$. This element is a scalar multiple of z: dividing by $\prod_i q_i$ gives z. This shows that z is a root of the monic polynomial

$$t^{\prod_i q_i} - \prod_i x_i^{p_i \sum_{j \neq i} q_j}$$

and therefore integral.

Remark 1.11.9. Note that if $k[\sigma \cap M]$ and $k[\langle (x_i)_{\mathbb{N}} \rangle]$ do not necessarily have the same fraction field, so the former is not necessarily the integral closure of the latter.

Take for example the ring inclusion $k[x^2] \subseteq k[x]$. Both are integrally closed but $k[x]$ is integral over $k[x^2]$.

Theorem 1.11.10. *The fan in Figure 8 is the normalized blow-up of $\mathbb{A}^2$ in the ideal $(X^{a_1} Y^{b_1}, X^{a_2} Y^{b_2})$.*

Proof. We show that the standard affine opens of this toric variety are isomorphic to those of the normalization of Definition 1.11.2 and that we glue along the same open.

The affine opens of the blow-up are the spectra of the affine rings

$$k\left[X, Y, \frac{X^{a_1}Y^{b_1}}{X^{a_2}Y^{b_2}}\right] = k[X, Y, X^{a_1-a_2}Y^{b_1-b_2}],$$
$$k\left[X, Y, \frac{X^{a_2}Y^{b_2}}{X^{a_1}Y^{b_1}}\right] = k[X, Y, X^{a_2-a_1}Y^{b_2-b_1}].$$

We claim that normalizing gives us exactly the desired rings corresponding to the cones, glued along the same opens.

We prove the statement for the cone σ. The other argument will work exactly the same. The ring $k[X, Y, X^{a_1-a_2}Y^{b_1-b_2}]$ is given by the monoid ring of the monoid drawn in Figure 9.

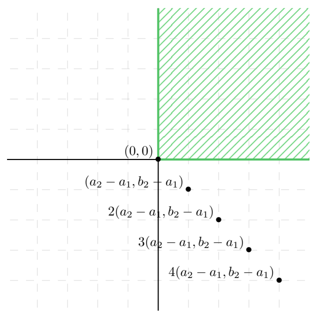


Figure 9: The monoid corresponding to the affine of the blow-up.

We claim that the normalization of this ring is exactly the desired cone: We need to show that all points of the cone correspond to monomials that are integral over this ring and that they have the same fraction field.

The first part is immediate from Lemma 1.11.8. At least one point of each ray is in the cone. Therefore, the ring $k[S_\sigma]$ is integral over the above ring.

To see the fraction fields are the same we simply note that both have fraction field $k(X, Y)$.